Figure 2 shows the motion of the object for $S^u = \{u^{[r]}, p_{y1} = 0.7, p_{y2} = 0.3\} \neq S_0^u$, S_0^v $[\gamma = 13.5 > \rho^\circ(t_*, x_*)]$.

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THREE-DIMENSIONAL MOTION OF A MATERIAL POINT†

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The two-dimensional motion of a material point over the active portion of its trajectory can be generalized in a natural way to three dimensions. Corresponding to the traditional flight plane, to which the trajectory of motion is confined in three dimensions, we have a set of flight surfaces obtained from it by bending. The three-dimensional system of differential equations governing the motion of a material point splits into a two-dimensional system, which describes the motion in the flight surface, and a system of ordinary differential equations, which describes the bending of the surface. By solving this system of equations one can determine by analytical means how the velocity and coordinate vectors over the active portion of the trajectory depend on its three-dimensional distortion. The results obtained may be used to analyse the three-dimensional motion of a material point, to select trajectories in space and to control the three-dimensional motion of the centre of mass over the active portions. In some cases one can actually derive analytical expressions for solutions to boundary-value and extremal problems associated with the three-dimensional motion of a material point.

1. THE BASIS TRIHEDRON AND THE DIFFERENTIAL EQUATIONS OF ITS ROTATION

WE SHALL be concerned with the three-dimensional motion of a material point over the active portion of its trajectory about a single attractive centre. A physical example of such a motion is that

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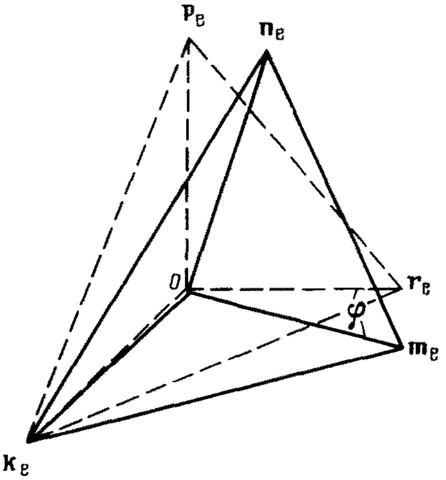

FIG. 1.

of a spacecraft putting an artificial satellite into orbit and manoeuvring in near-terrestrial space as it goes through non-coplanar orbits. This three-dimensional motion is described by the following system of equations:

$$d\mathbf{v}/dt = -\mu\mathbf{r}r^{-3} + \mathbf{u}(t, \mathbf{v}, \mathbf{r}), \quad d\mathbf{r}/dt = \mathbf{v} \tag{1.1}$$

where $\mathbf{v}$ is the velocity vector, $\mathbf{r}$ is the coordinate vector, t is the time, μ is the gravitational coefficient and $\mathbf{u}(t, \mathbf{r}, \mathbf{v})$ is a non-central acceleration vector, which generally depends on the reactive, aerodynamic forces, the eccentricity of the gravitational field and other factors affecting the motion.

We shall assume that at the starting time t_0 we have $\mathbf{v}(t_0) = \mathbf{v}_0$, $\mathbf{r}(t_0) = \mathbf{r}_0$. The motion of the point will take place in the fixed plane through these vectors if the non-central acceleration vector lies in that plane. Let $\mathbf{m}_e$ and $\mathbf{n}_e$ be a pair of unit vectors of arbitrary directions in the flight plane, attached to the centre of mass. Let $\mathbf{k}_e$ be a unit vector from the centre of mass, orthogonal to the flight plane, in the direction of the angular momentum vector. The trihedron of the three vectors $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$ will be called the basis trihedron. In the two-dimensional motion of a material point the position of the basis trihedron, and hence that of the vectors $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$, remains unchanged.

The position of the point on its trajectory is specified in terms of the angular distance φ, reckoned in the flight plane from the vector $\mathbf{m}_e$ in the counterclockwise sense (see Fig. 1). Then the unit coordinate vector $\mathbf{r}_e$ and the unit vector along the transversal $\mathbf{p}_e$ may be written as

$$\mathbf{r}_e = \mathbf{m}_e \cos\varphi + \mathbf{n}_e \sin\varphi, \quad \mathbf{p}_e = -\mathbf{m}_e \sin\varphi + \mathbf{n}_e \cos\varphi \tag{1.2}$$

The vectors $\mathbf{r}_e$, $\mathbf{p}_e$, $\mathbf{k}_e$ form what we call the moving trihedron.

If the projection of the non-central acceleration vector on the unit angular momentum vector, $u_k = (\mathbf{u}, \mathbf{k}_e)$, is not identically zero, the motion of the material point will be not two- but three-dimensional. The instantaneous flight plane and basis trihedron $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$ will rotate. The moving trihedron $\mathbf{r}_e$, $\mathbf{p}_e$, $\mathbf{k}_e$ will also rotate, but about a moving axis. In this three-dimensional motion of the basis and moving trihedrons the position of the unit radius-vector $\mathbf{r}_e$ and transversal vector $\mathbf{p}_e$ will obey the same formulas (1.2). The angle φ, which we call the three-dimensional angular distance, will be defined, as in two-dimensional motion, as the integral of the instantaneous angular velocity of revolution of the point over the orbit:

$$\varphi' = kr^{-2}, \quad \varphi = \varphi_0 + \int_{t_0}^{t_k} \varphi'(\tau)\, d\tau \tag{1.3}$$

where k and r are the moduli of the angular momentum and the coordinate vectors, respectively. The three-dimensional angular distance φ will be reckoned in the instantaneous flight plane from the vector $\mathbf{m}_e$ in the counterclockwise sense. Under these assumptions the three-dimensional rotation of the basis trihedron $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$ obeys the following system of differential equations:

$$d\mathbf{m}_e/dt=\psi'\mathbf{r}_e\times\mathbf{m}_e, \quad d\mathbf{n}_e/dt=\psi'\mathbf{r}_e\times\mathbf{n}_e$$

$$d\mathbf{k}_e/dt=\psi'\mathbf{r}_e\times\mathbf{k}_e \tag{1.4}$$

$$\psi'=u_k rk^{-1}=u_q \operatorname{tg}\gamma rk^{-1} \tag{1.5}$$

where u_q is the projection of the non-central acceleration on the instantaneous flight plane and γ is its inclination to that plane. The initial conditions for integrating system (1.4) in three dimensions are the same as in two dimensions.

We will prove that the basis trihedron $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$ does indeed rotate as described by Eqs (1.4). The kinematic equation $d(r\mathbf{r}_e)/dt=\mathbf{v}$, the equation of the angular momentum $d(k\mathbf{k}_e)/dt=r\mathbf{r}_e\times\mathbf{u}$ and the orthogonality relation $\mathbf{k}_e\times\mathbf{r}_e=\mathbf{p}_e$ imply a system of differential equations governing the rotation of the moving trihedron:

$$d\mathbf{r}_e/dt=\varphi'\mathbf{p}_e, \quad d\mathbf{p}_e/dt=-\varphi'\mathbf{r}_e+\psi'\mathbf{k}_e$$

$$d\mathbf{k}_e/dt=-\psi'\mathbf{p}_e \tag{1.6}$$

Differentiating Eqs (1.2) and using (1.5), (1.6) and (1.3), we obtain the equations of rotation of the basis trihedron (1.4), from which it follows that during the flight the trihedron $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$ rotates as a rigid body about the vector $\mathbf{r}_e$ at an instantaneous angular velocity ψ'. The same is true of the moving trihedron $\mathbf{r}_e$, $\mathbf{p}_e$, $\mathbf{k}_e$, with instantaneous angular velocity

$$\chi'=\varphi'\mathbf{k}_e+\psi'\mathbf{r}_e \tag{1.7}$$

These results hold true if the angular momentum vector does not vanish at the beginning, end or during the motion of the material point. If it does vanish, the unit angular momentum vector $\mathbf{k}_e$ is undefined and the angular velocity of rotation of the basis trihedron ψ' tends to infinity, as may be deduced from formula (1.5). This may be avoided if we confine our attention to trajectories on which the angular momentum vector k does not vanish. Then, however, our results would be inapplicable in such an important case as the launching of a spacecraft from a planet's surface. It is therefore desirable to eliminate this exception; this can be done using the fact that the unit angular momentum vector $\mathbf{k}_e$ is differentiable and the angular velocity of rotation of the basis trihedron ψ' is bounded in the neighbourhood of points where $\mathbf{k}=0$.

Let us consider the launching of a spacecraft from a planetary surface, in three cases. In the first case the ship moves vertically, along the radius-vector $\mathbf{v}\|\mathbf{r}_e$, $\mathbf{u}\|\mathbf{r}_e$. In the second case, it lifts vertically for a time t, and then turns instantaneously according to the pitching of the reactive acceleration vector. At time $t-0$, therefore, $\mathbf{v}\|\mathbf{r}_e$, $\mathbf{u}\|\mathbf{r}_e$, $t+0$ $\mathbf{v}\|\mathbf{r}_e$, $\mathbf{u}\nparallel\mathbf{r}_e$. In the third case we stipulate that $\mathbf{v}\|\mathbf{r}_e$, $\mathbf{u}\|\mathbf{r}_e$ at time $t+0$, but the angular velocity of rotation of the reactive acceleration changes suddenly from zero to $\omega\nparallel\mathbf{r}_e$,

In all three cases the vector $\mathbf{m}_e$ of the basis trihedron points in the direction of $\mathbf{r}_e$. In the first case the direction of $\mathbf{k}_e$ may be chosen at will in a plane orthogonal to $\mathbf{r}_e$, but the angular velocity of rotation of the basis trihedron must vanish identically. It can be shown that in the second and third cases, at the instant $t+0$,

$$\mathbf{k}_e=(\mathbf{r}_e\times\mathbf{u}_e)\,(1-(\mathbf{r}_e,\ \mathbf{u}_e)^2)^{-1/2}$$

$$\mathbf{k}_e=(\mathbf{r}_e\times(\omega\times\mathbf{r}_e))\,((\omega,\ \omega)-(\omega,\ \mathbf{r}_e)^2)^{-1/2} \tag{1.8}$$

The inclination of the reactive acceleration vector to the instantaneous flight plane will be chosen as in formula (1.5). Since the basis trihedron is rotating at a bounded velocity ψ', it follows that $\gamma=0$ at time $t+0$, so the reactive acceleration vector lies in the instantaneous flight plane. If the angular momentum vector $\mathbf{k}$ vanishes at an interior point of the trajectory then, applying the same formulas (1.8), we obtain the unit angular momentum vector $\mathbf{k}_e$ at time $t+0$.

2. THE THREE-DIMENSIONAL MOTION OF A MATERIAL POINT IN THE FLIGHT SURFACE. GENERATING AND GENERATED TRAJECTORIES

The unperturbed motion of a material point in a central field may be represented by velocities and coordinates in a fixed flight plane and by the vector $\mathbf{k}_e$, which describes the motion of the plane. The motion may be decoupled in the same way if there is a non-central acceleration, but it lies in the flight plane. We will show that a similar decoupling is possible for a three-dimensional orientation of the non-central acceleration.

Resolve the velocity, coordinate and non-central acceleration vectors into their components with respect to the basis vectors $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$:

$$\mathbf{v}=v_m\mathbf{m}_e+v_n\mathbf{n}_e, \quad \mathbf{r}=r_m\mathbf{m}_e+r_n\mathbf{n}_e \tag{2.1}$$

$$\mathbf{u}=u_m\mathbf{m}_e+u_n\mathbf{n}_e+u_k\mathbf{k}_e \tag{2.2}$$

Any three-dimensional motion of a material point about a single attracting centre is described by the following two-dimensional system of equations:

$$\begin{gathered} dv_m/dt=-\mu r_m(r_m^2+r_n^2)^{-3/2}+u_m \\ dv_n/dt=-\mu r_n(r_m^2+r_n^2)^{-3/2}+u_n \\ dr_m/dt=v_m, \quad dr_n/dt=v_n \\ v_{m0}=(\mathbf{v}_0, \mathbf{m}_{e0}), \quad v_{n0}=(\mathbf{v}_0, \mathbf{n}_{e0}) \\ r_{m0}=(\mathbf{r}_0, \mathbf{m}_{e0}), \quad r_{n0}=(\mathbf{r}_0, \mathbf{n}_{e0}) \end{gathered} \tag{2.3}$$

together with the equations of rotation (1.4) of the basis trihedron and relations (2.1). To prove this, insert (2.1) into (1.1) and use Eqs (1.4).

In the general case, when the kinetic projection of the non-central acceleration $u_k(t)$ is not identically zero, Eqs (2.3) describe the motion of the point in a certain flight surface.

We now introduce the notions of generating and generated trajectories. Let us assume that the motion of our material point in a central field takes place under the influence of a reactive acceleration, given by the flight-time functions

$$u_m=w_m^{\bullet}(t), \quad u_n=w_n^{\bullet}(t), \quad u_k=w_k^{\bullet}(t)$$

A trajectory obtained by integrating the equations of motion (1.1) on the assumption that the kinetic projection of the reactive acceleration $w_k^{\bullet}(t)$ is identically zero is called a generating trajectory. A trajectory obtained by integrating the same system of differential equations with the same initial conditions, but on the assumption that $w_k^{\bullet}(t)$ is not identically zero, is called a generated trajectory. For example, the two-dimensional trajectory described when an artificial satellite is placed in orbit, when the sighting azimuth at the starting point is 90°, is a generating trajectory. The three-dimensional trajectories described when artificial satellites are placed in orbit at other inclinations, with the same start and sighting azimuth, are generated trajectories. It is obvious that the basis trihedron on a generating trajectory is fixed, but on a generated trajectory it rotates as described by Eqs (1.4).

A generating trajectory $v_m(t)$, $v_n(t)$, $r_m(t)$, $r_n(t)$ is obtained by integrating the two-dimensional system of equations (2.3), which is independent of the kinetic projection of the non-central acceleration. If one knows the kinetic projection of the non-central acceleration $w_k^{\bullet}(t)$ and the velocities and coordinates v_m, v_n, r_m, r_n along a generating trajectory, then by integrating the equations of the basis trihedron (1.4) one can determine the current position of the vectors $\mathbf{m}_e(t)$, $\mathbf{n}_e(t)$, $\mathbf{k}_e(t)$, and subsequently, using formulas (2.1), calculate the velocity and coordinate vectors for any generated trajectory.

The class of generated trajectories associated with a given generating trajectory possesses very important properties: at any instant of time, material points moving along all generated trajectories (and, of course, along the generating trajectory) are at the same distance from the centre of mass, and the velocity vectors have the same magnitudes and are inclined at the same angle to the vertical. This is easily proved by using formulas (2.1) to construct the scalar products $(\mathbf{v}, \mathbf{v})$, $(\mathbf{v}, \mathbf{r})$, $(\mathbf{r}, \mathbf{r})$. It follows from Eqs (1.3) that at any given time the three-dimensional angular distances φ for the generating trajectory and all generated trajectories are equal.

If the velocity and coordinate vectors $\mathbf{v}$, $\mathbf{r}$ and the basis vectors $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$ along a generated trajectory are known, one can use (2.1) to determine the coordinates on the generating trajectory without integrating the velocity vectors. If the basis vectors $\mathbf{m}_e$ and $\mathbf{n}_e$ are known on two generated trajectories, then

$$\mathbf{v}_2=\mathbf{m}_{e2}(\mathbf{m}_{e1}, \mathbf{v}_1)+\mathbf{n}_{e2}(\mathbf{n}_{e1}, \mathbf{v}_1)$$

$$\mathbf{r}_2=\mathbf{m}_{e2}(\mathbf{m}_{e1}, \mathbf{r}_1)+\mathbf{n}_{e2}(\mathbf{n}_{e1}, \mathbf{r}_1) \tag{2.4}$$

A flight plane containing a generating trajectory will be called a generating plane. A flight surface in which a generated trajectory is situated will be called a generated surface. It is the envelope of the instantaneous flight planes $(\mathbf{k}_e(t), \mathbf{r}) = 0$ and is a ruled surface generated by the three-dimensional curve $\mathbf{r}_e(t)$:

$$\mathbf{r}(t, r)=r\mathbf{r}_e(t) \tag{2.5}$$

Since the tangent planes remain unchanged as r varies, this flight surface is developable, conic and may be developed. The generating plane is also a developed flight surface and the two-dimensional equations (2.3) describe the motion of a material point on this surface. The three-dimensional angular distance on a generated trajectory corresponds to the usual angular distance on the developed flight surface. Plot the coordinates of a generating trajectory on a sheet of paper, as well as the velocity vectors, suitably scaled. Folding the paper along radii, we obtain all possible generated trajectories and generated flight surfaces.

Our results for the case in which the material point is moving in a central field under the exclusive influence of reactive acceleration may be generalized to motion in a non-central field in the presence of an atmosphere or other perturbing factors. However, for the generated trajectory to be independent of them, we require the non-central acceleration vector to be regulated in such a way that the components $u_m(t)$ and $u_n(t)$ are given functions of the flight time.

Let us consider how our results can be used to select spacecraft trajectories for different inclinations of terrestrial satellite orbits. Usually, for each inclination of the satellite orbit one chooses a sighting plane that passes through the starting point, and within this plane, a program for the rotation of the reactive pitching acceleration vector. If one uses the results of Secs 1 and 2, a single generating trajectory $v_m(t)$, $v_n(t)$, $r_m(t)$, $r_n(t)$ must be chosen for a certain range of satellite orbits. This trajectory is defined in terms of a program for the pitch ϑ and the projection $w_k^{\bullet}(t)$ of the reactive acceleration in the instantaneous flight plane. The inclination γ of the reactive acceleration vector to the instantaneous flight plane must be chosen in accordance with the required inclination of the satellite orbit.

For generated trajectories the reactive acceleration and the consumed characteristic velocity must increase in accordance with the relations

$$w_2^{\bullet} = w_1^{\bullet}/\cos\gamma, \quad w_2 = \int_{t_0}^{t_k} w_1^{\bullet}(\tau)/\cos\gamma(\tau)\, d\tau$$

It follows from these formulas that at small angles γ the consumption is also not large.

3. SPLITTING OF THE GENERAL THREE-DIMENSIONAL MOTION OF A MATERIAL POINT INTO INDEPENDENT MOTIONS: THE RIGID FLIGHT SURFACE AND THE MOTION OF THE MATERIAL POINT ON IT

Let the variable of integration be the three-dimensional angular distance φ. The two-dimensional equations of motion of the material point along a generating trajectory and the equations of rotation of the basis and moving trihedrons in this case are

$$dv_m/d\varphi = -\mu k^{-1}\cos\varphi + r^2k^{-1}u_m,$$
$$dv_n/d\varphi = -\mu k^{-1}\sin\varphi + r^2k^{-1}u_n \tag{3.1}$$

$$dr/d\varphi = (v_m\cos\varphi + v_n\sin\varphi)r^2k^{-1}, \quad dt/d\varphi = r^2k^{-1}$$
$$d\mathbf{m}_e/d\varphi = \psi^{\cdot}\mathbf{r}_e\times\mathbf{m}_e, \quad d\mathbf{n}_e/d\varphi = \psi^{\cdot}\mathbf{r}_e\times\mathbf{n}_e \tag{3.2}$$
$$d\mathbf{k}_e/d\varphi = \psi^{\cdot}\mathbf{r}_e\times\mathbf{k}_e$$

$$d\mathbf{r}_e/d\varphi = \mathbf{p}_e, \quad d\mathbf{p}_e/d\varphi = -\mathbf{r}_e + \psi^{\cdot}\mathbf{k}_e$$
$$d\mathbf{k}_e/d\varphi = -\psi^{\cdot}\mathbf{p}_e. \tag{3.3}$$

Here

$$\psi^{\cdot} = \psi'/\varphi' = u_kr^3k^{-2} = u_q\,\mathrm{tg}\,\gamma r^3k^{-2} \tag{3.4}$$

is the relative angular velocity of rotation of the basis trihedron.

The solution of Eqs (3.1)–(3.3) as functions of φ will possess all the properties of the solution as functions of time. Generating and generated trajectories are defined in the same way. Equations (3.2) and (3.3), which describe the rotation of the basis and moving trihedrons, respectively, will acquire qualitatively new properties if it is not the kinetic projection u_k of the non-central acceleration that is given but the relative angular velocity of rotation of the basis trihedron, as a function of the three-dimensional angular distance, $\psi'(\varphi)$. The kinetic projection of the non-central acceleration is defined from (3.4), and therefore

$$u_k = k^2r^{-3}\psi^{\cdot}(\varphi) \tag{3.5}$$

Equations (3.2) and (3.3), which describe the rotation of the basis and moving trihedrons, are converted in this case to linear equations, which depend only on the relative angular velocity $\psi'(\varphi)$ of rotation of the basis trihedron and not at all on the generating trajectory. Their solutions—the unit vectors of the basis and moving trihedrons $\mathbf{m}_e$, $\mathbf{n}_e$, $\mathbf{k}_e$, $\mathbf{r}_e$, $\mathbf{p}_e$—depend only on the three-dimensional angular distance and relative angular velocity of rotation $\psi'(\varphi)$ of the basis trihedron.

As shown in Sec. 2, in that case, when the variable of integration is the time t, different generating trajectories determine different generated flight surfaces. If the variable of integration is the three-dimensional angular distance and the relative velocity of rotation of the basis trihedron is a function of this alone, then the generated conical flight surface

$$\mathbf{r}(r, \varphi) = r\mathbf{r}_e(\varphi) \tag{3.6}$$

is the same for all generated trajectories. We shall then say that the flight surface is rigid.

Let us consider the set of trajectories of the moving material point: high, low, with high or low flight speeds, affected by different non-central accelerations. If these trajectories are initially on the same radius-vector, in the same instantaneous plane, and the kinetic projection of the non-central acceleration during the motion is selected in accordance with condition (3.5), they will all lie on the same rigid flight surface. Hence the general problem of a material point moving in three dimensions under the influence of a non-central point moving in three dimensions under the influence of a non-central acceleration splits into two independent problems: to determine: (1) a rigid flight surface, and (2) the motion of the point on that surface.

Consider the unit sphere about the centre of mass, which cuts the rigid conical flight surface. All material points whose trajectories satisfy the conditions listed above will describe the same track, that is, the intersection of the conical surface with the sphere. The unit vectors of the transversal and the angular momentum will also describe the same track.

It was shown in Secs 1 and 2 how the results could be derived by using the basis and moving trihedrons. To determine them, however, it was necessary to integrate the differential equations governing their rotation. Another way to get the trihedrons is to integrate the ordinary three-dimensional equations of motion for the material point itself, i.e. system (1.1), and the differential equations (1.3) of the three-dimensional angular distance. Let us assume that by integrating these equations we have found the velocity and coordinate vectors $\mathbf{v}$, $\mathbf{r}$ and the three-dimensional angular distance φ. Knowing these quantities we determine the vectors of the moving trihedron $\mathbf{r}_e$, $\mathbf{p}_e$, $\mathbf{k}_e$ and then, from (2.1), the vectors $\mathbf{m}_e$ and $\mathbf{n}_e$ of the basis trihedron. The trihedrons thus determined may be used to choose trajectories in space and to design a control system for the motion of the mass centre.

Here is an example. Let us suppose that the relative angular velocity of rotation $\psi'(\varphi)$ of the basis trihedron is given. Integrating Eqs (1.1) and (1.3), we determine $v(t)$, $r(t)$ and $\varphi(t)$ for each instant of time. Knowing φ, we can determine $\psi^{\cdot}(\varphi)$ for the same time. Using formulas (3.4), we then determine γ and thence the direction of the non-central acceleration vector necessary to reorient the trajectory in space as required.

4. ANALYTICAL DEFINITION OF THE SPACE TRAJECTORY. ANALYTICAL SOLUTION OF BOUNDARY-VALUE AND EXTREMAL PROBLEMS

When the relative velocity of rotation of the basis trihedron $\psi^{\cdot}(\varphi)$ is a constant, the system of differential equations (3.3), describing the rotation of the moving trihedron, becomes linear with constant coefficients and can therefore be solved analytically. We introduce a rectangular system of coordinates $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, which coincides at the starting time with the vectors $\mathbf{r}_e$, $\mathbf{p}_e$, $\mathbf{k}_e$. In this system of coordinates the solution of system (3.3) is

$$\begin{aligned}
\mathbf{r}_e &= \frac{1}{2}\begin{Vmatrix} 1+\cos 2\alpha \\ 0 \\ \sin 2\alpha \end{Vmatrix} + \frac{1}{2}\begin{Vmatrix} 1-\cos 2\alpha \\ 0 \\ -\sin 2\alpha \end{Vmatrix}\cos\chi + \begin{Vmatrix} 0 \\ \sin\alpha \\ 0 \end{Vmatrix}\sin\chi \\
\mathbf{p}_e &= \begin{Vmatrix} -\sin\alpha \\ 0 \\ \cos\alpha \end{Vmatrix}\sin\chi + \begin{Vmatrix} 0 \\ 1 \\ 0 \end{Vmatrix}\cos\chi \\
\mathbf{k}_e &= \frac{1}{2}\begin{Vmatrix} \sin 2\alpha \\ 0 \\ 1-\cos 2\alpha \end{Vmatrix} + \frac{1}{2}\begin{Vmatrix} -\sin 2\alpha \\ 0 \\ 1+\cos 2\alpha \end{Vmatrix}\cos\chi + \begin{Vmatrix} 0 \\ -\cos\alpha \\ 0 \end{Vmatrix}\sin\chi
\end{aligned} \tag{4.1}$$

where

$$\operatorname{ctg}\alpha=\psi^{\cdot}, \quad \varphi=\chi\sin\alpha \tag{4.2}$$

It follows from these formulas that the rigid flight surface is a circular cone with aperture angle α and axis $(\cos\alpha,\ 0,\ \sin\alpha)$ in the initial plane $\mathbf{r}_{e0}$, $\mathbf{k}_{e0}$. The moving trihedron rotates at a constant velocity around the axis, χ being its angle of rotation. The vectors $\mathbf{r}_e$, $\mathbf{p}_e$, $\mathbf{k}_e$ of the moving trihedron describe circles on the unit sphere at angles $\pi/2-\alpha$, 0, α to the equatorial plane.

The generating trajectory, as a function of the three-dimensional angular distance, may be specified not in rectangular coordinates but in terms of the distance from the centre of mass $r(\varphi)$, the radial and transversal components $v_r(\varphi)$, $v_p(\varphi)$ of the velocity vector and the flight time. In that case one has following relations:

$$\mathbf{r}=r(\varphi)\mathbf{r}_e(\varphi), \quad \mathbf{v}=v_r(\varphi)\mathbf{r}_e(\varphi)+v_p(\varphi)\mathbf{p}_e(\varphi) \tag{4.3}$$

which are similar to (2.1). Substituting the known values of the vectors $\mathbf{r}_e$, $\mathbf{p}_e$ from (4.1) into (4.3), we obtain analytical expressions for the components of the velocity and coordinate vectors:

$$\mathbf{r} = \frac{r(\varphi)}{2} \begin{Vmatrix} 1 + \cos 2\alpha + (1 - \cos 2\alpha)\cos\chi \\ 2\sin\alpha\sin\chi \\ \sin 2\alpha - \sin 2\alpha\cos\chi \end{Vmatrix} \tag{4.4}$$

$$\mathbf{v} = \frac{1}{2} \begin{Vmatrix} ((1 - \cos 2\alpha)\cos\chi + 1 + \cos 2\alpha) v_r - 2\sin\alpha\sin\chi v_p \\ 2\sin\alpha\sin\chi v_r + 2\cos\chi v_p \\ (-\sin 2\alpha\cos\chi + \sin 2\alpha) v_r + 2\cos\alpha\sin\chi v_p \end{Vmatrix}$$

We shall show that the methods developed here will yield analytical expressions for solutions of a three-dimensional boundary value problem. Let us assume that the problem is concerned with the rotation of the orbital plane through a given angle β, Then it must be true that

$$(\mathbf{k}_{e0}, \mathbf{k}_e) = \cos\beta$$

Substituting the vector $\mathbf{k}_e$ from (4.1) into this formula and carrying out some algebra, we get an equation

$$\cos\alpha \sin\chi/2 = \sin\beta/2$$

Determining the angle α from (4.1), we determine the relative angular velocity of rotation $\psi^{\cdot}$ of the basis trihedron and the value of the angular distance φ, after which $\psi^{\cdot}$ must be equated to zero. The analytical solution of this boundary value problem does not depend in any way on the generating trajectory. Therefore, once the parameters $\psi^{\cdot}$ and φ have been chosen on any three-dimensional trajectory, the instantaneous flight plane will turn through the given angle β, provided the kinetic projection u_k of the non-central acceleration is given by (3.5).

We will now show that the methods developed in this paper sometimes make it possible to obtain analytical solutions of extremal problems. Let us assume that the active portion is sufficiently short. Then, if the unit vector $\mathbf{r}_e$ does not vary significantly during flight along the active portion, an examination of Eqs (1.4) shows that the angle of rotation β of the orbital plane is approximately equal to the angle of rotation ψ of the basis trihedron. Consider the problem of minimizing the consumption of characteristic velocity w for a given rotation ψ of the basis trihedron:

$$w = \int_{t_0}^{t_k} ((w_q^{\cdot})^2 + (w_k^{\cdot})^2)^{1/2}\, d\tau$$

$$\psi = \int_{t_0}^{t_k} w_k^{\cdot} r k^{-1}\, d\tau$$

We will assume that a generating trajectory is given. Then, varying the conditional functional $w + \psi/\lambda$ as a function of the kinetic projection $w_k^{\cdot}$ of the reactive acceleration and performing the necessary algebra, we obtain an analytical expression for the inclination of the reactive acceleration to the instantaneous flight plane:

$$\operatorname{tg}\gamma = \lambda(v_p^2 - \lambda^2)^{-1/2} \tag{4.5}$$

The constant λ must satisfy the relationship

$$\psi = \lambda \int_{t_0}^{t_k} v_p^{-1}(v_p^2 - \lambda^2)^{-1/2} w_q^{\cdot}\, d\tau \tag{4.6}$$

When satellites are placed in orbit nowadays, variation of the flight plane is generally achieved by specifying the sighting plane at the start; as the point continues its flight in the sighting plane, the direction of the jet thrust turns in accordance with the pitch. The simple quasi-optimal method (4.5), (4.6) for specifying the jet thrust direction may somewhat reduce the consumption of characteristic velocity necessary to place a satellite in orbit.

Translated by D.L.